Robustly modelling the scale and shape dynamics of stock return distributions*

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Abstract

We explore time variation in the shape of the conditional return distribution using a model of multiple quantiles. We propose a joint model of scale (proxied by the interquartile range) and other quantiles standardised by the scale. The model allows us to estimate the scale and shape of the distribution, which are both time varying, in one step. We find that, once we capture the dynamics of the scale effectively, the time variation in the shape allows a simpler interpretation.

Keywords: Dynamic multivariate quantile model; Return decomposition; Robust methods; CAViaR model

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1 Introduction

The conditional distribution of asset returns has been widely studied and modelled using a range of techniques. This has led to a large amount of evidence that returns of most assets display not only time-dependence in variance, but also in higher moments. In this paper, we model time variation in the conditional return distribution using a semi-parametric approach based on joint models of multiple quantiles.

Using a joint model of conditional quantiles, we separately specify dynamics for a common time-varying scale and the quantiles standardised by this time-varying scale. The joint model is estimated in a single step. Our proposed approach naturally extends GARCH and stochastic volatility models and allows for a richer analysis of the conditional distribution via quantiles.

An important advantage of our approach is that the scale acts as a common factor that builds dependence across quantiles in a parsimonious manner. In turn, this makes it easier to estimate the model with the quantiles in their standardised form.

There is a long history of research on the shape of the return distribution, which is of central importance to asset pricing and risk management. Modellers have developed many techniques to capture the fat-tailed and often skewed nature of return distributions (Bates 2006). By introducing the conditionally heteroskedastic class of time series models, Engle (1982) separated the shape of the underlying return surprise from its time-varying and serially dependent scale. An entire literature has followed, introducing more complicated dynamics for the scale and different distributions for the underlying shocks. The GARCH(1,1) with standard normal shocks (Bollerslev 1986) captures a great deal of dependence in the conditional variance, but requires modification to better fit the tails of many return distributions (French et al. 1987, Bollerslev 1987, Hansen & Lunde 2005). Firstly, a leverage effect (Black 1976) can be incorporated to allow asymmetric responses of volatility to positive and negative returns (Schwert 1989, Nelson 1991, Glosten et al. 1993, Engle & Ng 1993); or, secondly, alternative distributions can be applied for the underlying shocks (Bollerslev 1987, McNeil & Frey 2000). Similarly, models with fat tails and long memory in volatility (Ding et al. 1993) can be obtained through fractionally integrated GARCH models (Baillie et al. 1996) or component models (Ding & Granger 1996, Engle & Lee

1999). Semiparametric and nonparametric models for the conditional return distribution have also been explored. For example, Engle & Gonzalez-Rivera (1991) and Kalli et al. (2014) consider using GARCH models, whereas Jensen & Maheu (2010) and Delatola & Griffin (2011) use stochastic volatility models.

Outside the discrete time GARCH framework, continuous time stochastic volatility models are also able to generate fat-tailed return distributions with leverage effects. These models can also incorporate features such as jumps in returns and volatility to better explain the dynamics of returns and to improve their applicability to problems such as option pricing; see *e.g.* Chernov et al. (2003) and Bates (2006) for an overview.

Despite the above improvements to GARCH or stochastic volatility models, there remains residual difficulty with fitting the tails of the return distribution. For instance, Gallant et al. (1991) show that the standardised error term has a time-varying conditional distribution. As it is difficult to model the entire distribution, modelling efforts beyond the variance have focused on higher moments such as skewness and kurtosis. Hansen (1994) proposed an explicit model of time-varying higher moments, while Harvey & Siddique (1999) applied an auto-regressive conditional skewness model to returns. Backus et al. (1997) made use of a Gram Charlier expansion around the normal to back out time-varying skewness and kurtosis from the option smirk, while Christoffersen et al. (2013) utilised the inverse Gaussian distribution to obtain time-varying skewness in an option pricing model.

Rather than rely on estimators of higher moments based on averages of past realised moments, Kim & White (2004) proposed the use of quantile based estimators. They showed through simulations how empirical moment based estimators are significantly more sensitive to outliers. Effectively, they question existing beliefs about the extent and the time variation of skewness and kurtosis.

Following Kim & White (2004), White et al. (2010) proposed joint models of conditional quantiles to obtain robust estimates of conditional skewness and kurtosis.[1] This approach

[1]The approach of estimating moments from quantiles has also been applied by others, including Taylor (2005) who estimates volatility from an interquantile range, and Xiao & Koenker (2009) who estimate GARCH models in a two-step procedure. Ghysels et al. (2011) examine the higher moments to form portfolios, while Coroneo & Veredas (2012) to study their interdependence over time.

develops CAViaR models (Engle & Manganelli 2004) which work well in estimating particular probability levels of the conditional distribution such as the Value at Risk (VaR), (see *e.g.* Engle & Manganelli 2001, Kuester et al. 2006, Gerlach et al. 2011). They also offer modelling flexibility as the conditional quantiles can respond to observations without assuming a parametric return distribution. However, multiple quantiles cannot be satisfactory estimated using separate application of single quantile models at different probability levels since these estimates do not guarantee monotonicity of the quantile function, which is often called the crossing problem (see *e.g.* Chernozhukov & Galichon 2008, Chernozhukov et al. 2010, Gourieroux & Jasiak 2008, for explanations and possible solutions). Thus, it is useful to jointly model quantiles and avoid quantiles crossing in the estimation procedure.

The specification of the multi-quantile model (MQ CAViaR) of White et al. (2010) takes the form of a vector autoregression, which they find awkward to estimate. We propose an alternative model for multiple quantiles which builds on the intuition that most of the variation in the shape of the conditional return distribution should be captured by a time-varying volatility process. Our approach builds a dependence between the quantiles via the common scale factor (rather than the linear functions of a vector autoregression). This model is more parsimonious than the MQ CAViaR model and achieves a robust scale and shape decomposition of the conditional return distribution. This allows simpler comparison to commonly used models for financial time series which directly model the conditional variance.

Previous papers where the scale information is separately obtained and then used to enhance quantile models, include Jeon & Taylor (2013) and Chen & Gerlach (2014). Jeon & Taylor (2013) model VaR by incorporating option implied volatility in a CAViaR model. Chen & Gerlach (2014) use intra-day data to capture volatility and tail risk for estimating expected shortfall (ES) in an auto-regressive expectile model (Taylor 2008). Our approach differs by only using the information in the daily returns to model both the scale and the shape dynamics in a single estimation stage.

The paper is organised as follows. The following section provides a brief background on the MQ CAViaR model. Section 3 presents the construction of our model and discusses alternative forms for the time evolution of the scale and conditional quantiles. Section 4 describes the estimation of the model. Applications of the models to stock index and stock price data are discussed in Section 5. To ensure that we are capturing the dynamics well, the models are evaluated and compared in terms of both in-sample fit and out-of-sample forecasting criteria. Section 6 concludes.

2 Background: MQ CAViaR

For a random variable X with distribution function $F(\cdot)$, we define q_θ to be the quantile of X at probability level θ if $F(q_\theta) = \theta$. The MQ CAViaR model of White et al. (2010) specifies a general process for the quantiles of conditional distributions of the return series $r_1, r_2, \ldots, r_T$ at probability levels $\theta_1, \theta_2, \ldots, \theta_K$. We define $\mathbf{q_t} = (q_{\theta_1,t}, \ldots, q_{\theta_K,t})'$ where $q_{\theta,t}$ is the conditional quantile of r_t at probability level θ (this name emphasises the conditional quantiles' difference to the quantiles of the marginal distribution of r_t). The MQ CAViaR model assumes that

$$\mathbf{q_t} = u + \sum_{i=1}^{M} \beta_i \, \mathbf{q_{t-i}} + \sum_{j=1}^{L} \gamma_j \, l_j \left(r_1, \ldots, r_{t-1}\right), \tag{1}$$

where M and L are the orders of the model, u and γ_j are K-dimensional vectors, β_i is a $(K \times K)$-dimensional matrix and $l_j\left(r_1, \ldots, r_{t-1}\right)$ is a function of the history of the return series.

A model with $M = 1$, $L = 1$ and $l_1(r_1, \ldots, r_{t-1}) = |r_{t-1}|$ is used in the analysis of data by White et al. (2010) leading to the form

$$\mathbf{q_t} = u + \beta_1 \, \mathbf{q_{t-1}} + \gamma_1 |r_{t-1}|. \tag{2}$$

Each conditional quantile is assumed to depend linearly on its first lag, on the first lags of the other conditional quantiles, and on the absolute value of the previous return.

The MQ CAViaR is a generalisation to multiple conditional quantiles of the CAViaR model of a single conditional quantile $q_{\theta,t}$ (Engle & Manganelli 2004) which assumes that

$$q_{\theta,t} = u_\theta + \sum_{i=1}^{M} \beta_{\theta,i} q_{\theta,t-i} + \sum_{j=1}^{L} \gamma_{\theta,j} \, l_j \left(r_1, \ldots, r_{t-1}\right), \tag{3}$$

where u_θ, $\beta_{\theta,i}$ and $\gamma_{\theta,j}$ are scalars. The model with $M = 1$, $L = 1$ and $l_1(r_1, \ldots, r_{t-1}) = |r_{t-1}|$ is called the Symmetric Absolute Value (SAV) CAViaR model and has the form

$$q_{\theta,t} = u_\theta + \beta_{\theta,1} q_{\theta,t-1} + \gamma_{\theta,1} |r_{t-1}|. \tag{4}$$

The model in (2) simplifies to this model for all probability levels if β_1 is a diagonal matrix.

3 An alternative approach: Scale and shape decomposition

A reasonable goal for a multi-quantile time series model is to gain a better understanding of the evolution of the conditional quantiles (and, more generally, the conditional distribution), and to improve the prediction of individual conditional quantiles using information from other conditional quantiles. To guide us about the form of the relationship between conditional quantiles without making any parametric assumptions about the conditional return distribution, we first examine several separately estimated conditional quantiles. The conditional quantiles with probability levels 0.01, 0.05, 0.25, 0.50, 0.75, 0.95, and 0.99 were separately estimated using a univariate Component CAViaR model (see *e.g.* Mitrodima & Oberoi 2015). The data consists of S&P500 index returns obtained from CRSP for the period January $01, 2002$ to December $31, 2014$. By plotting the separately estimated conditional quantiles together in Figure 1, we observe the time-variation in the scale of the distribution. The conditional quantiles increase and decrease together (as might be expected), but they also appear to do so proportionately to each other.

To make this point clearer, we plot the differences between adjacent conditional quantiles in Figure 2. The differences between adjacent conditional quantiles tend to follow very similar patterns over time. This suggests that a substantial part of the joint dynamics of the conditional quantiles could be simply captured in a multi-quantile time series model by introducing a common scale process (a common scale parameter is often assumed in models of returns data such as a GARCH model). If we want estimates of the MQ CAViaR which avoid the conditional quantiles crossing, we will need to place constraints on the cross-autoregressive terms. Since these constraints need to be applied to the fitted conditional quantiles at each time point, the set of allowable parameter combinations will become

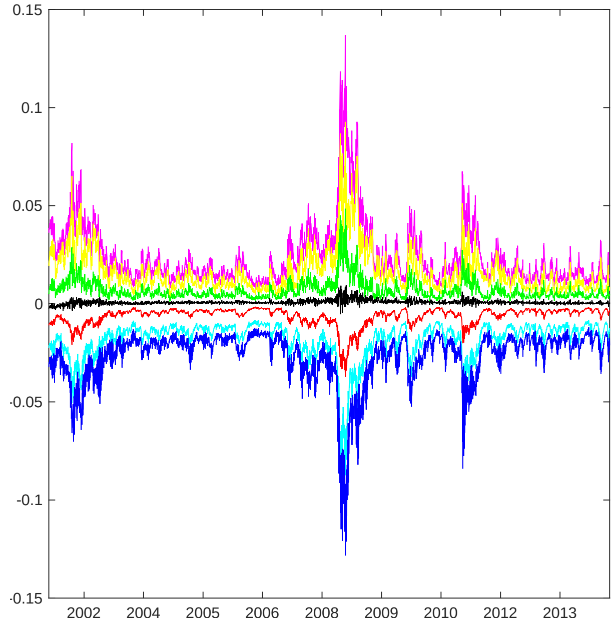


Figure 1: Separately estimated conditional quantiles for S&P500

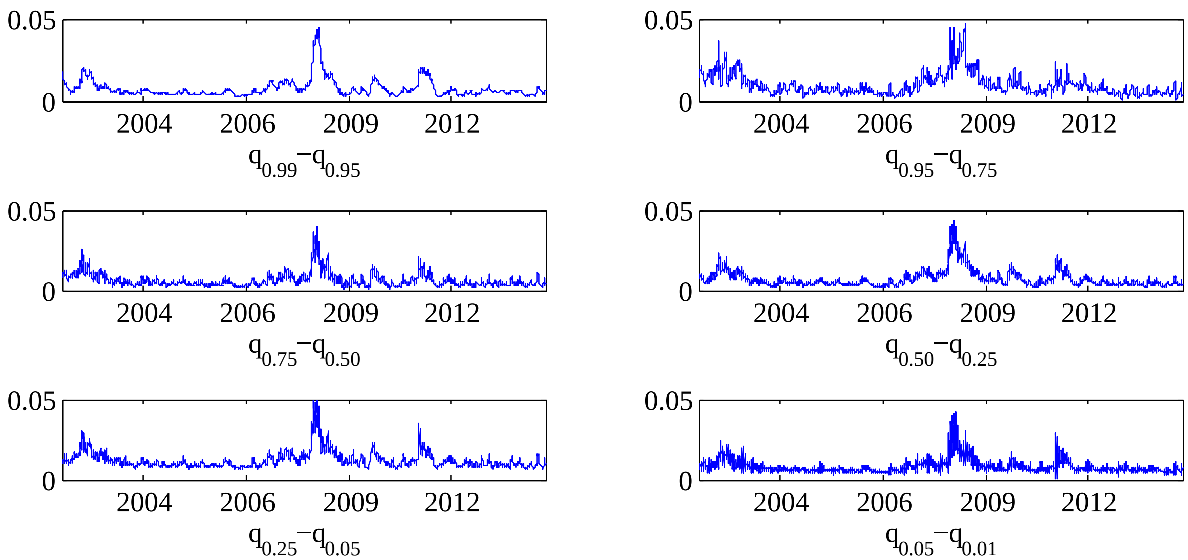


Figure 2: Differences between adjacent conditional quantiles for S&P500

smaller as the observed time series becomes longer. Consider modelling two conditional quantiles using the MQ CAViaR model in (2) with

$$u = \begin{pmatrix} 0.01 \\ 0.05 \end{pmatrix}, \beta_1 = \begin{pmatrix} \beta_{0.01,0.01} & \beta_{0.01,0.05} \\ \beta_{0.05,0.01} & \beta_{0.05,0.05} \end{pmatrix} \text{ and } \gamma_1 = \begin{pmatrix} \gamma_{0.01} \\ \gamma_{0.05} \end{pmatrix}.$$

For a given $\beta_{0.01,0.01}$, the range of values of γ_1 that avoid crossings (i.e. $q_{0.05,t} < q_{0.01,t}$) becomes increasingly restricted as $\beta_{0.01,0.05}$ increases. Given the usual range of variation of $|r_t|$, this exacerbates the well known problem of allocating appropriate weights to past observations of returns.

Our modelling approach assumes that $r_t = s_t z_t$, where s_t is time-varying scale and z_t is a standardised return which follows a time-varying distribution (many standard models for returns assume a time-invariant distribution for z_t). We define s_t to be a robust scale measure defined in terms of the conditional quantiles of r_t (such as the conditional interquartile range) and assume that a further $K-1$ conditional quantiles of the standardised returns follow independent CAViaR models. Suppose that $s_t = q_{\theta_1,t} - q_{\theta_2,t}$, where $\theta_1 = 1-\alpha$ and $\theta_2 = \alpha$ for $\alpha < 1/2$ and $\theta_3, \ldots, \theta_K$ are $K-2$ further probability levels then our model is

$$\begin{aligned} s_t = q_{\theta_1,t} - q_{\theta_2,t} &= u^{(s)} + \sum_{i=1}^{M} \beta_i^{(s)} s_{t-i} + \sum_{j=1}^{L} \gamma_j^{(s)}\, l_j\left(r_1, \ldots, r_{t-1}\right) \\ q_{\theta_k,t}^{(z)} &= u_{\theta_k}^{(z)} + \sum_{i=1}^{M} \beta_{\theta_k,i}^{(z)} q_{\theta_k,t-i}^{(z)} + \sum_{j=1}^{L} \gamma_{\theta_k,j}^{(z)}\, l_j\left(r_1, \ldots, r_{t-1}\right), \quad k = 2, \ldots, K, \end{aligned} \tag{5}$$

where M and L are the orders of the model, $u^{(s)}$, $\beta_i^{(s)}$, and $\gamma_i^{(s)}$ are scalars, u and $\gamma_j^{(z)}$ are $(K-1)$-dimensional vector, $\beta_i^{(z)}$ is a $((K-1) \times (K-1))$-dimensional matrix and $l_j\left(r_1, \ldots, r_{t-1}\right)$ represents a function of the history of the return series. We choose to use the conditional interquartile range which arises if $\alpha = 0.25$ as our robust scale measure which seems reasonable if there is less time-variation in the body of the distribution than the tails of the distribution. In our application, we have found that the results do not change with smaller values of α.

The model can be expressed in terms of the conditional quantiles of the return distribution at probability levels $\theta_1, \theta_2, \ldots, \theta_K$ using the transformation $q_{\theta_1,t} = q_{\theta_2,t} + s_t$ and

$q_{\theta_j,t} = s_t q^{(z)}_{\theta_j,t}$ for $j = 1, 2, \ldots, K$ leading to the following model for $q_{\theta_1,t}, q_{\theta_2,t}, \ldots, q_{\theta_K,t}$,

$$
\begin{aligned}
s_t &= u^{(s)} + \sum_{i=1}^{M} \beta_i^{(s)} s_{t-i} + \sum_{j=1}^{L} \gamma_j^{(s)}\, l_j\left(r_1, \ldots, r_{t-1}\right) \\
q_{\theta_1,t} &= q_{\theta_2,t} + s_t \\
q_{\theta_k,t} &= s_t \left(u^{(z)}_{\theta_k} + \sum_{i=1}^{M} \beta^{(z)}_{\theta_k,i} \frac{q_{\theta_k,t-i}}{s_{t-i}} + \sum_{j=1}^{L} \gamma^{(z)}_{\theta_k,j}\, l_j\left(\frac{r_1}{s_1}, \ldots, \frac{r_{t-1}}{s_{t-1}}\right)\right), \quad k = 2, \ldots, K.
\end{aligned}
$$

Choosing $M = 1$, $L = 1$, $\theta_1 = 0.25$, $\theta_2 = 0.75$ and $l_1(r_1, \ldots, r_{t-1}) = |r_{t-1}|$ leads to a generalisation of the SAV CAViaR model, which we define to be the *J-SAV-IQR model* which has the form

$$
\begin{aligned}
s_t = q_{\theta_1,t} - q_{\theta_2,t} &= u^{(s)} + \beta_1^{(s)} s_{t-1} + \gamma_1^{(s)} |r_{t-1}| \\
q_{\theta_k,t} &= s_t \left(u^{(z)}_{\theta_k} + \beta^{(z)}_{\theta_k,1} \frac{q_{\theta_k,t-1}}{s_{t-1}} + \gamma^{(z)}_{\theta_k,1} \frac{|r_{t-1}|}{s_{t-1}} \right), \quad k = 2, \ldots, K.
\end{aligned}
\tag{6}
$$

This model has some interesting features. It directly models the extent of time variation in the shape of the conditional return distribution after controlling for the scale. It allows us to robustly estimate the scale (without separately estimating a volatility model) and the conditional return distribution in a single estimation step. The quantile crossing problem can be avoided during estimation, without assuming an explicit model for the conditional return distribution. It is also more parsimonious than the MQ CAViaR model and, consequently, can potentially enjoy improved estimation and predictive performance. If we consider the MQ CAViaR model in (2) for K probability levels with $M = 1$ and $L = 1$, the number of parameters is $2K + K^2$. In contrast, when $M = 1$ and $L = 1$, the J-SAV-IQR model has $3K$ parameters and the difference becomes larger as K increases

While the SAV is one of the simplest dynamic structures proposed for CAViaR models, several variations of this model have been developed, (see *e.g.* Engle & Manganelli 2001, 2004, Kuester et al. 2006, Gerlach et al. 2011, Mitrodima & Oberoi 2015). Some of the features that these models account for include the leverage effect, long range dependence in the quantile dynamics, and a time-varying mean of the returns themselves. The Component Asymmetric Slope (C-AS) CAViaR model from Mitrodima & Oberoi (2015) incorporates all the above features by allowing the mean of the quantile process u to be time-varying, introducing long range dependence in the same spirit as Engle & Lee (1999). To avoid the model becoming over-parameterised and following our findings from Figure 2, we consider only giving the scale process this structure leading to a scale process of the form

$$\begin{aligned} s_t = q_{\theta_1,t} - q_{\theta_2,t} &= u_t^{(s)} + \beta_1^{(s)} \left(s_{t-1} - u_{t-1}^{(s)} \right) + \gamma_1^{(s)} \, r_{t-1}^{+} + \gamma_2^{(s)} \, r_{t-1}^{-} \\ u_t^{(s)} &= \omega^{(s)} + \rho^{(s)} \, u_{t-1}^{(s)} + \gamma_3^{(s)} \, r_{t-1}, \end{aligned} \tag{7}$$

where $r^+ = \max(r, 0)$ and $r^- = -\min(r, 0)$. The modelling of the conditional quantiles of z retains the SAV CAViaR form in (6). We call this the *J-SAV-CASIQR model* and its full specification is

$$\begin{aligned} s_t = q_{\theta_1,t} - q_{\theta_2,t} &= u_t^{(s)} + \beta_1^{(s)} \left(s_{t-1} - u_{t-1}^{(s)} \right) + \gamma_1^{(s)} \, r_{t-1}^{+} + \gamma_2^{(s)} \, r_{t-1}^{-} \\ u_t^{(s)} &= \omega^{(s)} + \rho^{(s)} \, u_{t-1}^{(s)} + \gamma_3^{(s)} \, r_{t-1}, \\ q_{\theta_k,t} &= s_t \left(u_{\theta_k}^{(z)} + \beta_{\theta_k,1}^{(z)} \frac{q_{\theta_k,t-1}}{s_{t-1}} + \gamma_{\theta_k,1}^{(z)} \frac{|r_{t-1}|}{s_{t-1}} \right), \quad k = 2, \ldots, K. \end{aligned}$$

Notice that this model nests the J-SAV-IQR model in (6).

4 Estimation

The parameters of the J-SAV-IQR and J-SAV-CASIQR models with probability levels $\theta_1, \theta_2, \ldots, \theta_K$ can be estimated by applying the regression quantile criterion (Koenker & Bassett 1978, Engle & Manganelli 2004, Chernozhukov & Umantsev 2001), interpreted as a set of estimating equations (see *e.g.* Komunjer 2005, Komunjer & Vuong 2010, White et al. 2010). We write the vector of all parameters as Φ and use the Quasi-Maximum Likelihood Estimator (QMLE) $\hat{\Phi}$ which maximises the quasi-likelihood function

$$-\frac{1}{T} \sum_{t=1}^{T} \sum_{k=1}^{K} [\theta_k - I(r_t < q_{\theta_k,t}(\Phi))][r_t < q_{\theta_k,t}(\Phi)],$$

where T is the sample size and I is the indicator function. White et al. (2010) prove the consistency of the MQ CAViaR estimator under a set of general conditions. They also derive a central limit theorem and show how the covariance matrix of the parameters can be consistently estimated. Their methods can be used to prove the same results for our models under a similar set of assumptions.

The quasi-likelihood function can be highly multi-modal for multi-quantile time series models such as MQ CAViaR and the models developed in this paper. Therefore, calculation of the QMLE can be challenging. This problem is less pronounced with the J-SAV-IQR and J-SAV-CASIQR than the MQ CAViaR since there are less parameters to be estimated. However, estimates for each model can be sensitive to the estimation strategy used. For J-SAV-IQR, J-SAV-CASIQR and MQ CAViaR, we apply an estimation strategy proposed by Mitrodima et al. (2016). Separate optimisers of the QML are run for a set of initial parameter values (the Matlab optimiser "fminsearch" was used in our examples), and the outputted parameter value with the largest QML is reported as the estimate. The set of initial parameter values are chosen to be consistent with the data and to have a correct ordering for the quantiles. This procedure leads to improved estimates of the MQ CAViaR model relative to the original estimation procedure described in White et al. (2010), and is also particularly well suited to our model.

In the case of J-SAV-IQR model, the procedure begins by choosing a small set of possible values for $\beta^{(z)}_{\theta_k,i}$ and $\beta^{(s)}_i$ between 0.5 and 0.9. For instance, one may start off by setting all the $\beta^{(z)}_{\theta_k,i}$ at 0.5 and then at 0.7, and $\beta^{(s)}_i$ at 0.5 and 0.9. This initially gives four sets of starting β values. Associated with each set of β values, a set of five possible values for $\gamma^{(z)}_{\theta_k,i}$ or $\gamma^{(s)}_i$ is chosen. The possible values for $\gamma^{(z)}_{\theta_k,i}$ are $\{-0.2, -0.1, -0.02, -0.01, 0\}$ if $\theta_k < 0.5$ or $\{0, 0.01, 0.02, 0.1, 0.2\}$ if $\theta_k > 0.5$. For $\gamma^{(s)}_i$ and $\gamma^{(z)}_{0.5,i}$, both positive and negative values from the above sets are chosen. A starting value for $u^{(z)}_{\theta_k}$ can be chosen so that the unconditional mean of $q_{\theta_k,t}/s_t$ is equal to the corresponding quantile of a standard normal distribution (although, other distribution could be used). The initial value for $u^{(z)}_{\theta_k}$ is

$$u^{(z)}_{\theta_k} = (1 - \beta^{(z)}_{\theta_k,1})\overline{q_{\theta_k}} - \gamma^{(z)}_{\theta_k,1} E\left[\frac{|r_t|}{s_t}\right],$$

where $\overline{q_{\theta_k}}$ is the quantile of standard normal distribution at probability level θ_k. We also set $E\left[\frac{|r_t|}{s_t}\right] = \frac{1}{1.347}\sqrt{\frac{2}{\pi}}$. A similar strategy is applied to the IQR, whereby $u^{(z)}_t$ is chosen so that the unconditional mean of s_t is equal to the corresponding sample IQR. This procedure can easily be extended to the J-SAV-CASIQR model.

The procedure for finding initial values for the J-SAV-IQR and J-SAV-CASIQR models exploits the use of standardised quantiles but it can be extended to the MQ CAViaR model. In the case of MQ CAViaR model, we set the diagonal elements of the matrix β_1

and the elements of the vector γ_1 in the same way as for the scaled models above. All cross diagonal elements of β are initialised at 0. We then choose each element of the vector u so that the unconditional mean of each q_{θ_k} is equal to the corresponding empirical quantile estimated from the data. Our estimation approach is similar in spirit to the variance targeting strategy explained in Christoffersen (2003), but it is only used to initialise the optimisation, not as a constraint. Following Engle & Manganelli (2004), we initialise the quantile and interquantile range series at their empirical counterparts (using the first 300 observations).

5 Analysis of stock price and stock index data

We use data provided by CRSP. We analyse the S&P500 index, along with the stocks IBM, Exxon Mobil, and Intel Corporation (INTC). The complete data set consists of daily log returns from January $01, 2002$ to December $31, 2014$, with the last two years reserved for out-of-sample testing. We will initially describe inference over a ten year period from January 01, 2002 to understand the underlying dynamics of the return distribution and then discuss out-of-sample predictive performance over a two-year period using daily rolling estimates of the conditional quantiles *without* re-estimation of the model parameters. For comparison, we estimate the MQ CAViaR as well as our proposed scale-shape models J-SAV-IQR and J-SAV-CASIQR for the $0.01, 0.05, 0.25, 0.50, 0.75, 0.95$ and 0.99 probability levels.

5.1 Results

The estimated conditional quantiles and their values standardised by the conditional interquartile range for INTC and S&P500 are shown in Figures 3 and 4. A vertical line marks the end of the in-sample estimation period and the beginning of the out-of-sample period. The results show that there is substantial time-variation in the conditional quantiles captured by all models for both INTC and S&P500. The estimated standardised conditional quantiles for all models show much less variation over time and illustrate that a lot of the time-variation in the conditional quantiles can be captured through the con-

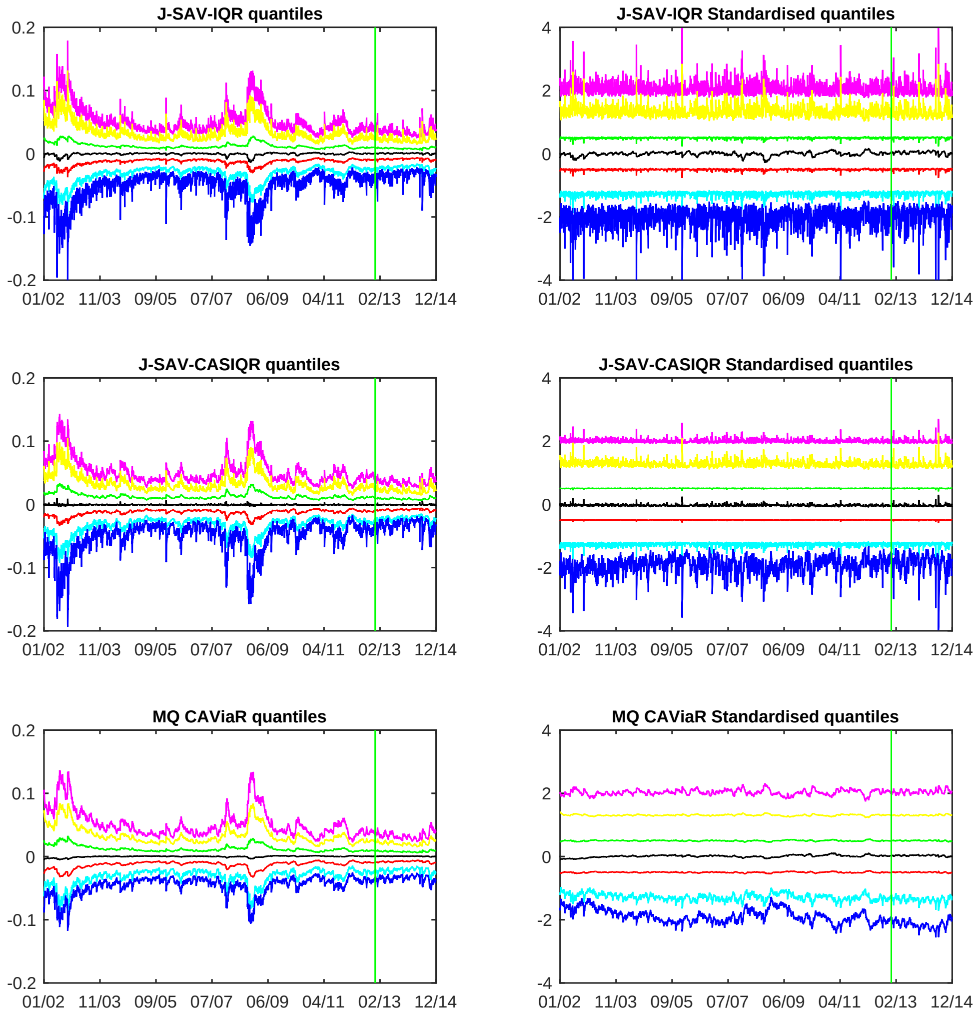


Figure 3: Quantile estimates for INTC

ditional interquartile range. The smoothness of the estimated standardised conditional quantiles varies with the probability level with the paths becoming rougher in the tails of the distribution. The estimated standardised conditional quantiles also show clear difference between the J-SAV-IQR and J-SAV-CASIQR models (which directly model the conditional interquartile range) and the MQ CAViaR model. The MQ CAViaR model has smoother estimates of all standardised conditional quantiles apart from the lower 1% point.

To better understand how the models respond to shocks, we plot the estimated conditional quantiles for INTC in the period from September 1, 2007 to August 31, 2009, covering

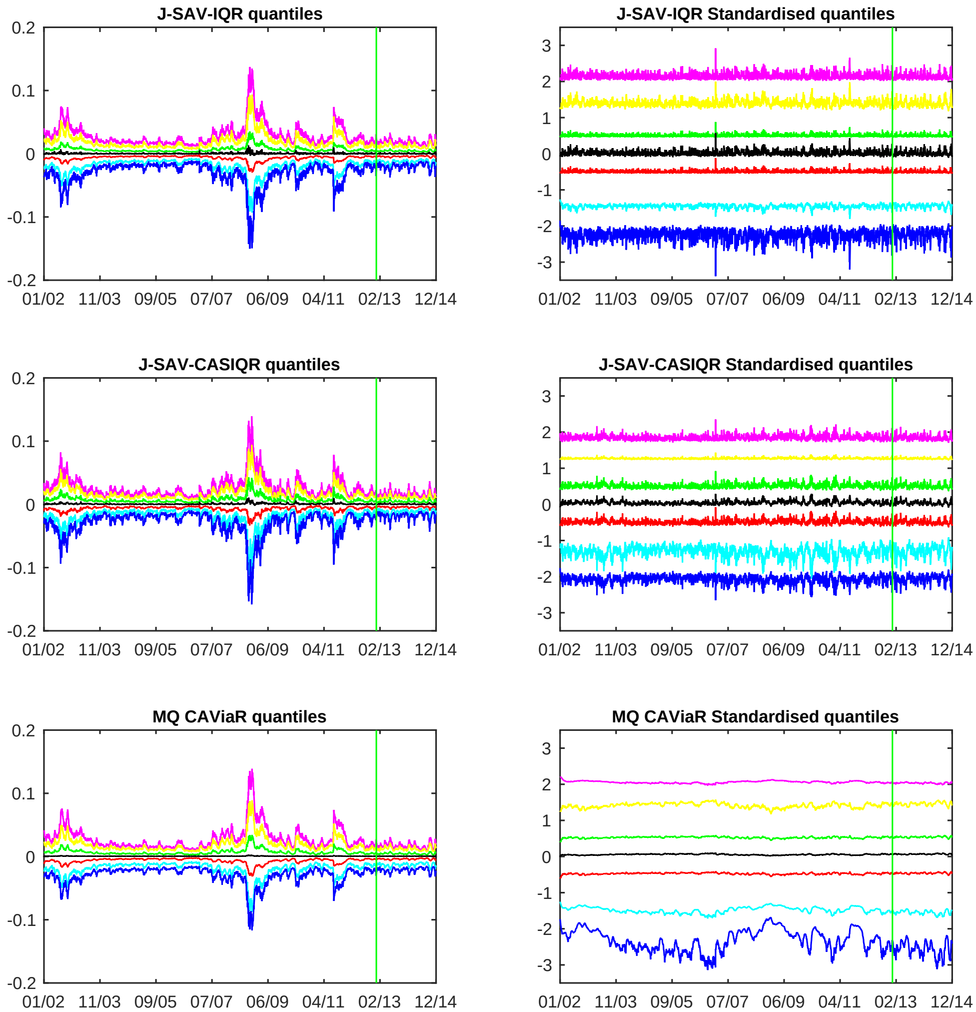


Figure 4: Quantile estimates for S&P500

the financial crisis. There are two clear shocks: one in early 2008 (as the subprime mortgage crisis spread) and another in September 2008 (due to the collapse of Lehman Brothers). In both cases, the J-SAV-CASIQR model has the largest response to shock followed by the J-SAV-IQR and MQ CAViaR models. For both shocks, there is a clear pattern of a temporary decrease in the 1% standardised quantile and increase in the 99% standardised quantile which indicates that the conditional return distribution becomes heavier tailed in these periods. This is consistent with our intuition about the conditional return distribution at times of market stress. The estimated standardised quantiles of the MQ CAViaR model

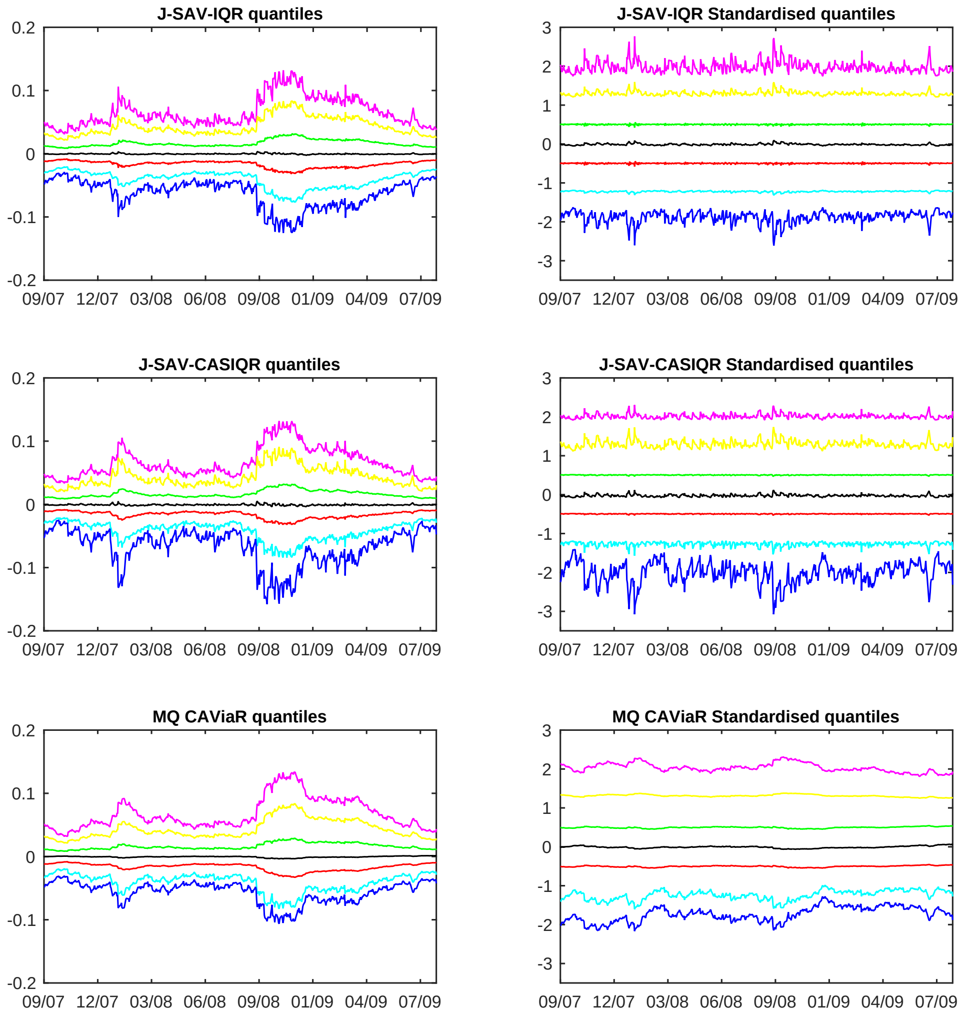


Figure 5: Quantile estimates for INTC from September 01, 2007, to August 31, 2009.

behave in a different way with a steady shift inwards of the 1% standardised quantile and much smaller increases in 99% standardised quantile. The J-SAV-IQR and J-SAV-CASIQR models show a much sharper response of the unstandardised conditional quantiles to the financial crisis in 2008.

The conditional quantiles estimates for INTC from from September 1, 2007 to August 31, 2009 using the J-SAV-IQR and MQ CAViaR models are compared to the conditional quantiles estimates from a GARCH(1,1) model with t-distributed innovations in Figure 6. The latter model assumes that the standardised conditional quantiles are constant over

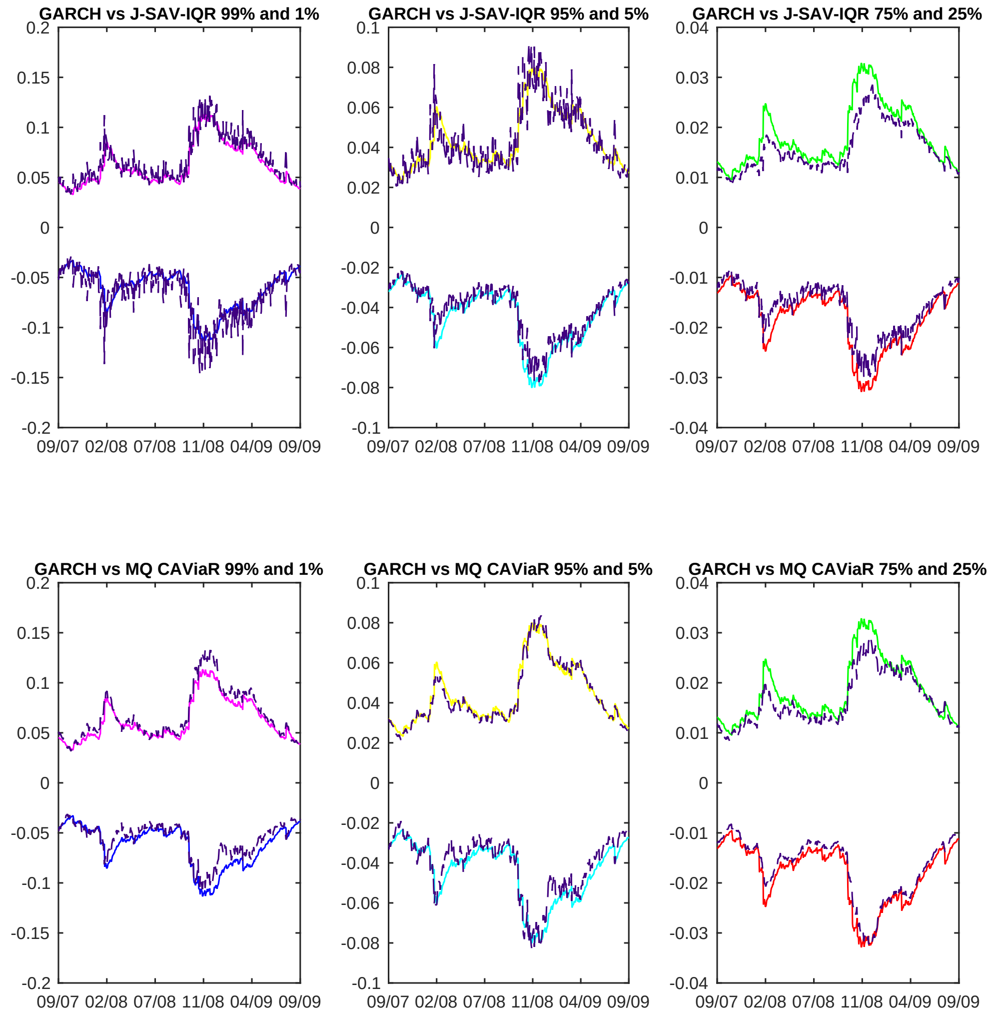


Figure 6: GARCH vs J-SAV-IQR (upper panel) and GARCH vs MQ CAViaR (lower panel) for INTC from September 01, 2007, to August 31, 2009.

time. The estimates of the conditional quantiles are similar apart from the periods following the two shocks. Following the shocks, the 75% and 25% conditional quantiles move further away from 0 with the GARCH model than the quantile based models. This is because the GARCH model cannot accommodate the heavier tails following a shock leading to overestimation of the volatility. This emphasises the importance of allowing the shape of the conditional distribution of returns to change over time.

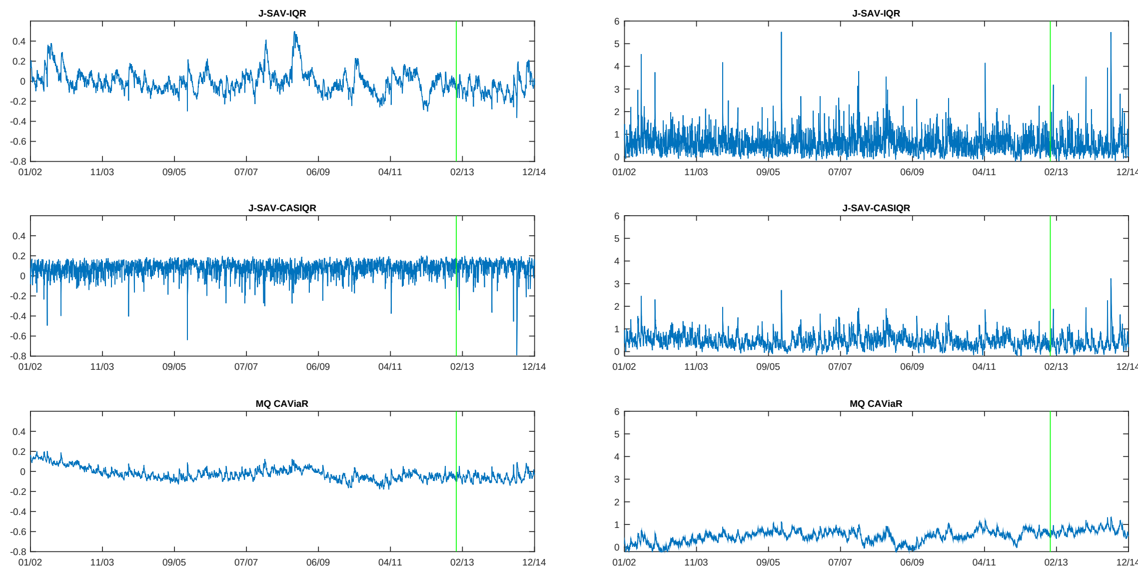
The estimates of the conditional skewness and kurtosis can be calculated from the estimated conditional quantiles using the following formulae suggested by White et al. (2010) and Kim & White (2004), who follow the statistics literature on robust estimators of higher moments. The daily conditional skewness is estimated as

$$\frac{q_{0.75,t} + q_{0.25,t} - 2 \times q_{0.50,t}}{q_{0.75,t} - q_{0.25,t}}$$

and the kurtosis as

$$\frac{q_{0.99,t} - q_{0.01,t}}{q_{0.75,t} - q_{0.25,t}} - 3.45.$$

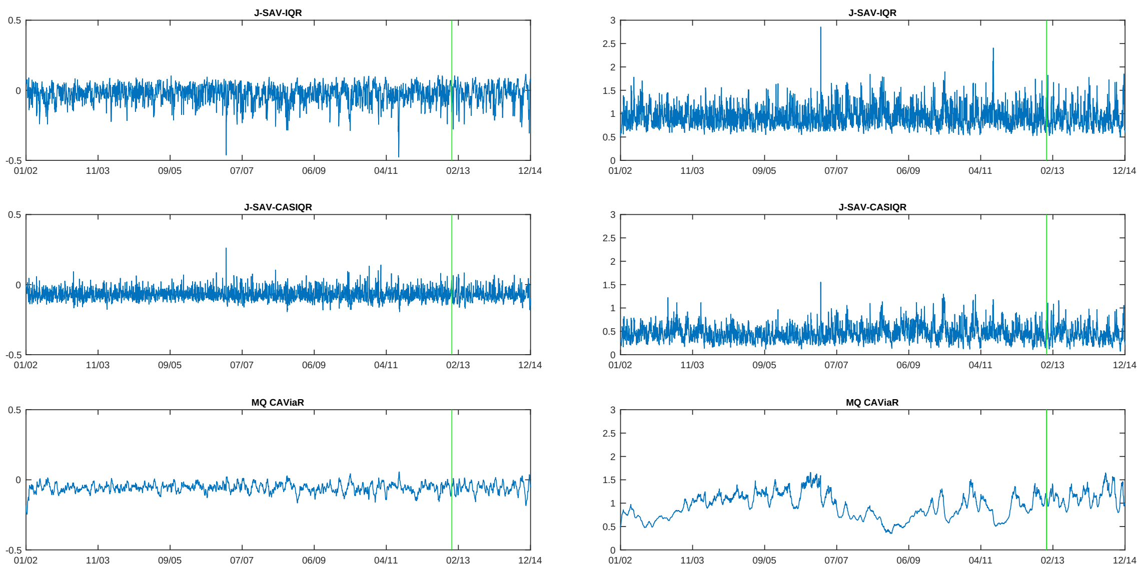
In Figures 7 and 8, we plot the series of the conditional skewness and kurtosis for INTC and S&P500 respectively, estimated by the three models. The conditional skewness results



(a) Skewnes (b) Kurtosis

Figure 7: Quantile-based measures of conditional skewness and kurtosis for INTC

for INTC using the J-SAV-IQR model (in Figure 7) show a clear pattern with periods of both positive and negative skewness. These estimates are consistent with what we might expect from a stock, with positive skewness in better times and negative skewness in times of stress. The MQ CAViaR model also shows a similar pattern over time but smooths out some of the features. The J-SAV-IQR and J-SAV-CASIQR also generate much greater range in the conditional kurtosis over time. In contrast, the estimated conditional skewness



(a) Skewnes

(b) Kurtosis

Figure 8: Quantile-based measures of conditional skewness and kurtosis for S&P500

for the S&P500 index is negative at all times but with a much wider range of values for the J-SAV-IQR and J-SAV-CASIQR models. Once again, the negative conditional skewness of the index is consistent with the stylised facts of returns data. Although the causes of this effect may be disputed, it is well known that indices display a more pronounced leverage effect than their constituent stocks individually. One explanation for this is that correlations between returns are higher in bad times than in good.

5.2 Coverage tests

One of the ways of evaluating the fit or the performance of the models is to compare their in-sample, and respectively, out-of-sample hit ratios (coverage). Of particular importance is the out-of-sample performance of the models in the tails of the distribution, as seen in Table 1. We report in- and out-of-sample hit ratios ($\frac{1}{T}\sum_{t=1}^{T} I(r_t < q_{\theta,t})$) for each of the joint quantile models for the three assets in turn.

Table 1: In- and out-of-sample hit ratios ($\frac{1}{T}\sum_{t=1}^{T} I(r_t < q_{\theta,t})$) for each of the joint quantile models

	S&P500							IBM						
θ	0.99	0.95	0.75	0.50	0.25	0.05	0.01	0.99	0.95	0.75	0.50	0.25	0.05	0.01
J-SAV-IQR														
In-Sample	0.992	0.947	0.747	0.478	0.242	0.052	0.011	0.990	0.947	0.751	0.494	0.245	0.049	0.011
Out-of-sample	0.996	0.966	0.700	0.424	0.210	0.048	0.008	0.990	0.966	0.760	0.518	0.250	0.050	0.014
J-SAV-CASIQR														
In-Sample	0.990	0.950	0.758	0.490	0.241	0.058	0.011	0.991	0.946	0.752	0.497	0.244	0.050	0.011
Out-of-sample	0.998	0.966	0.716	0.444	0.202	0.062	0.012	0.994	0.966	0.780	0.518	0.244	0.046	0.014
MQ CAViaR														
In-Sample	0.990	0.948	0.751	0.499	0.250	0.050	0.010	0.990	0.951	0.749	0.500	0.250	0.050	0.011
Out-of-sample	0.990	0.972	0.723	0.473	0.220	0.048	0.006	0.968	0.968	0.759	0.523	0.262	0.054	0.014
	Intel Corp							Exxon Mobil						
θ	0.99	0.95	0.75	0.50	0.25	0.05	0.01	0.99	0.95	0.75	0.50	0.25	0.05	0.01
J-SAV-IQR														
In-Sample	0.989	0.950	0.753	0.493	0.248	0.051	0.011	0.989	0.950	0.754	0.502	0.245	0.049	0.010
Out-of-sample	0.982	0.952	0.772	0.446	0.194	0.036	0.014	0.990	0.968	0.772	0.552	0.206	0.042	0.008
J-SAV-CASIQR														
In-Sample	0.990	0.951	0.755	0.486	0.246	0.051	0.011	0.989	0.955	0.758	0.502	0.252	0.053	0.010
Out-of-sample	0.986	0.962	0.776	0.438	0.188	0.034	0.012	0.992	0.974	0.810	0.548	0.214	0.040	0.008
MQ CAViaR														
In-Sample	0.990	0.950	0.749	0.502	0.251	0.049	0.010	0.990	0.950	0.750	0.497	0.250	0.050	0.010
Out-of-sample	0.980	0.942	0.755	0.467	0.194	0.028	0.012	0.992	0.966	0.761	0.543	0.256	0.038	0.008

Note: The sample ranges from January $01, 2002$, to December $31, 2014$, with the last two years being out-of-sample.

All models perform well overall, with the two new models matching the coverage of the MQ CAViaR model with far fewer parameters. On the other hand, none of the multiple quantile models seem to be able to capture potential short-term shifts in the location of the return distribution. As a result, the out-of-sample forecasts of the inner conditional quantiles appear to be biased. This may support the case for allowing the returns themselves to have some dependence, as suggested by Kuester et al. (2006). An alternative explanation is that our proxy for the scale does a good job of capturing the long-range dependence in scale, thereby doing a poor job of matching up the quartiles on an *unconditional* basis. The explanation one accepts is based on whether we consider the case of $S\&P500$, where the out-of-sample coverage appears to have shifted, or the other cases where one of the conditional quantiles is predicted fairly accurately, but the conditional interquartile range pushes the other quartile out on average.

While unconditional hit ratios are widely applied, there is a large literature proposing better tests of interval forecasts (see *e.g.* Christoffersen 1998, Christoffersen et al. 2001, Engle & Manganelli 2004, Komunjer & Giacomini 2005, Berkowitz et al. 2011, Gaglianone et al. 2011). Based on the testing literature, we need to examine whether the violation or hit sequence produced by the model has a predictable pattern. We apply three tests to each of the quantile series out-of-sample to ensure that the models' performance is comparable to, or better than, existing quantile forecasting models.

Following the suggestion of Berkowitz et al. (2011), we report the Dynamic Quantile (DQ) test of Engle & Manganelli (2004) in Table 2.[2] Overall, the models perform quite well on the tests compared to the models available in the existing literature. None of the four outer conditional quantiles are rejected in any of the models for any of the assets at the 5% significance level.[3]

However, the objective of this paper is not only to propose a more parsimonious and

[2]Other tests calculated but not reported include the unconditional coverage test and conditional coverage test from Christoffersen (1998). All results are available from the authors and will be placed in an online appendix.

[3]All the tests reported are univariate, i.e., testing one quantile at a time. The focus of this literature and model development has been on univariate specifications because of their use in forecasting extreme losses in finance and insurance. We have used these tests mainly for comparability and to demonstrate that they perform better or no worse than similar univariate models.

Table 2: DQ test p-values

	J-SAV-IQR	J-SAV-CASIQR	MQ CAViaR
S&P500			
θ			
0.99	0.9291	0.7839	0.0032
0.95	0.6625	0.5311	0.4002
0.75	0.0019	0.0641	0.0098
0.50	0.0012	0.0375	0.1295
0.25	0.0032	0.0407	0.0101
0.05	0.2300	0.3722	0.2098
0.01	0.8899	0.7012	0.9639
IBM			
θ			
0.99	0.9726	0.9467	0.9983
0.95	0.5695	0.2995	0.1436
0.75	0.3357	0.2977	0.1757
0.50	0.6919	0.2743	0.3682
0.25	0.4073	0.1780	0.3376
0.05	0.8707	0.4433	0.7463
0.01	0.9534	0.9505	0.9381
Intel Corp			
θ			
0.99	0.3839	0.1885	0.0259
0.95	0.4950	0.5342	0.2735
0.75	0.5963	0.4895	0.7152
0.50	0.2023	0.0046	0.2804
0.25	0.0905	0.0142	0.0395
0.05	0.6307	0.6365	0.3716
0.01	0.4599	0.9033	0.8870
Exxon Mobil			
θ			
0.99	0.9980	0.9863	0.9990
0.95	0.4371	0.0732	0.2616
0.75	0.2353	0.0602	0.1523
0.50	0.2117	0.2679	0.3995
0.25	0.1258	0.0835	0.0685
0.05	0.8918	0.7649	0.8080
0.01	0.9968	0.9984	0.9965

intuitive model for multiple conditional quantiles, but also to examine the implications for time variation in the shape of the underlying distribution.

6 Conclusion

In this paper, we have proposed a multi-quantile time series model for the conditional distribution of returns. The model is constructed by specifying a process for scale (which is taken to be the interquartile range) and the standardised conditional quantiles (which control the overall shape of the distribution). This mimics the standard approach to modelling financial time series, where the returns are modelled as the product of a scale and a random shock drawn from a time-invariant distribution (for example, GARCH or stochastic volatility modelling). Our approach builds a single model for several conditional quantiles jointly and so avoids the need for two-stage estimation. The decomposition also assists with the crossing problem, as does our estimation strategy of targeting the level of the quantile process when setting initial values.

A major benefit of this approach is that we have both a robust estimate of the time-varying scale of return distribution whilst also having time-varying estimates of the shape of the conditional distribution. This allows us to better understand where the largest changes in the conditional distribution of returns occur. Our results suggest that a robustly estimated scale can capture most of the dynamics of the conditional return distribution which leaves a relatively stable underlying shape that can be captured by fairly simple dynamics. Our model is more parsimonious than the MQ CAViaR since each standardised conditional quantile follows a CAViaR (rather than MQ CAViaR) model.

The estimated conditional return distribution offers useful insights into the dynamics of stock and index returns. While each quantile, when individually modelled, requires long memory and asymmetry to be captured adequately, this effect is subsumed in the scale dynamics. Once the scale dynamics are accounted for, the standardised tails are suitably represented by a very simple time series process, with very occasional spikes and a little structure still remaining. This supports the use of a sufficiently complicated volatility structure to model fat-tailed and skewed return distributions, but not necessarily the introduction of complicated distributional assumptions for the underlying shocks.

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Appendix: Additional Figures

This appendix contains figures corresponding to those reported in the paper, for the other assets studied. Figures A1 and A2 correspond to Figures 3 and 4. Figures A3, A4 and A5 correspond to Figure 5. Figures A6, A7 and A8 correspond to Figure 6. Figures A9 and A10 correspond to Figures 7 and 8.

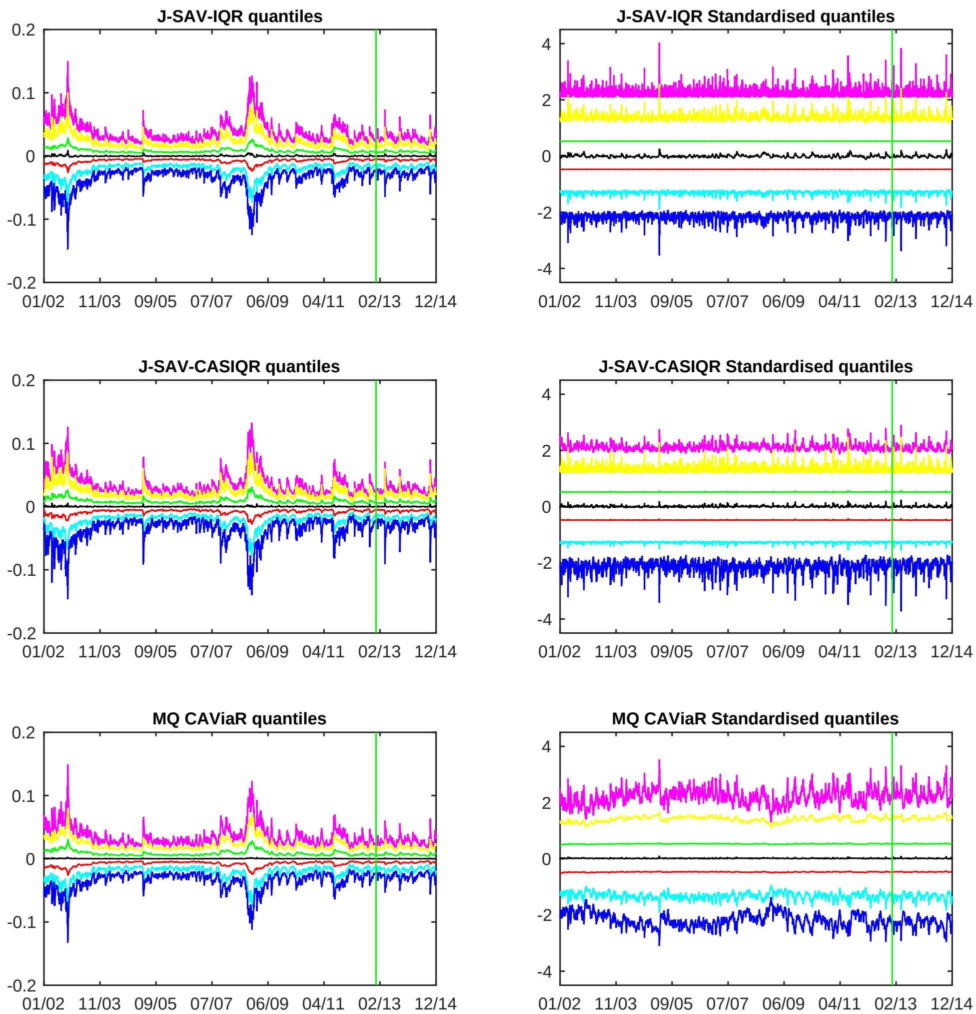


Figure A1: Quantile estimates for IBM

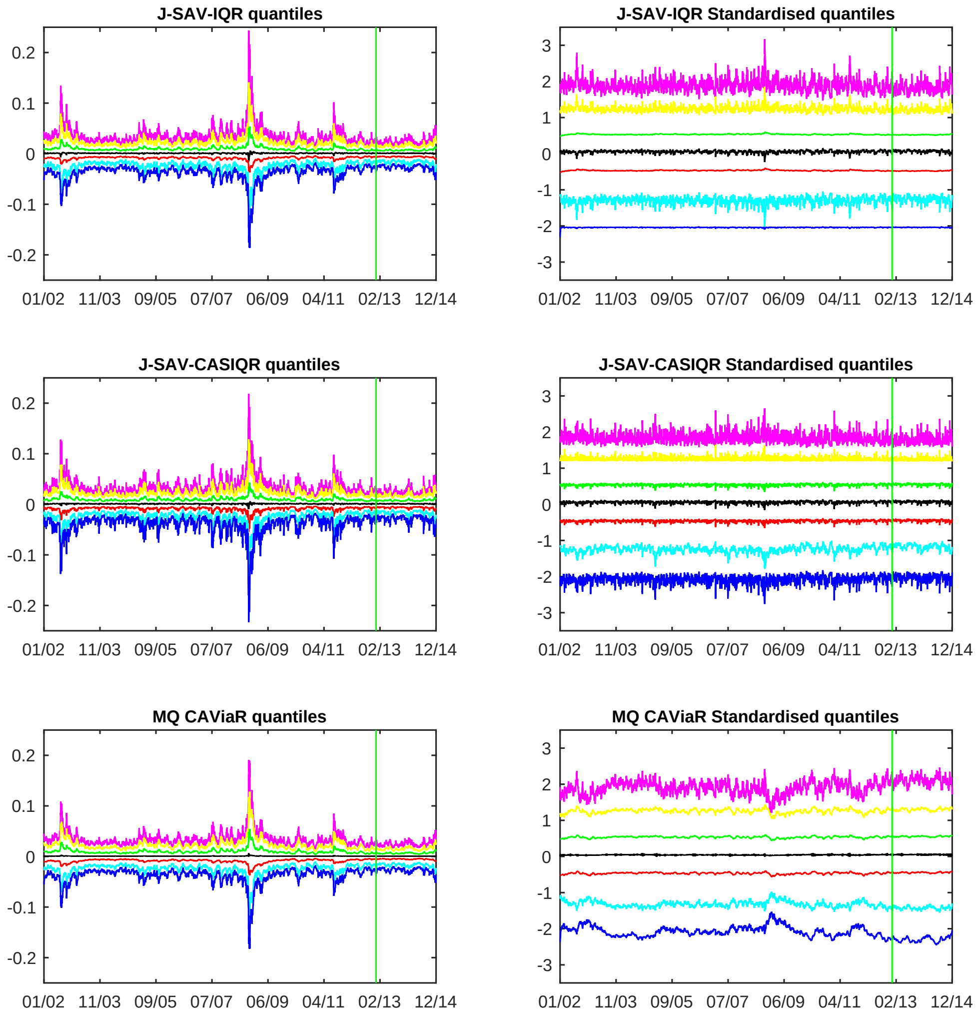


Figure A2: Quantile estimates for XOM

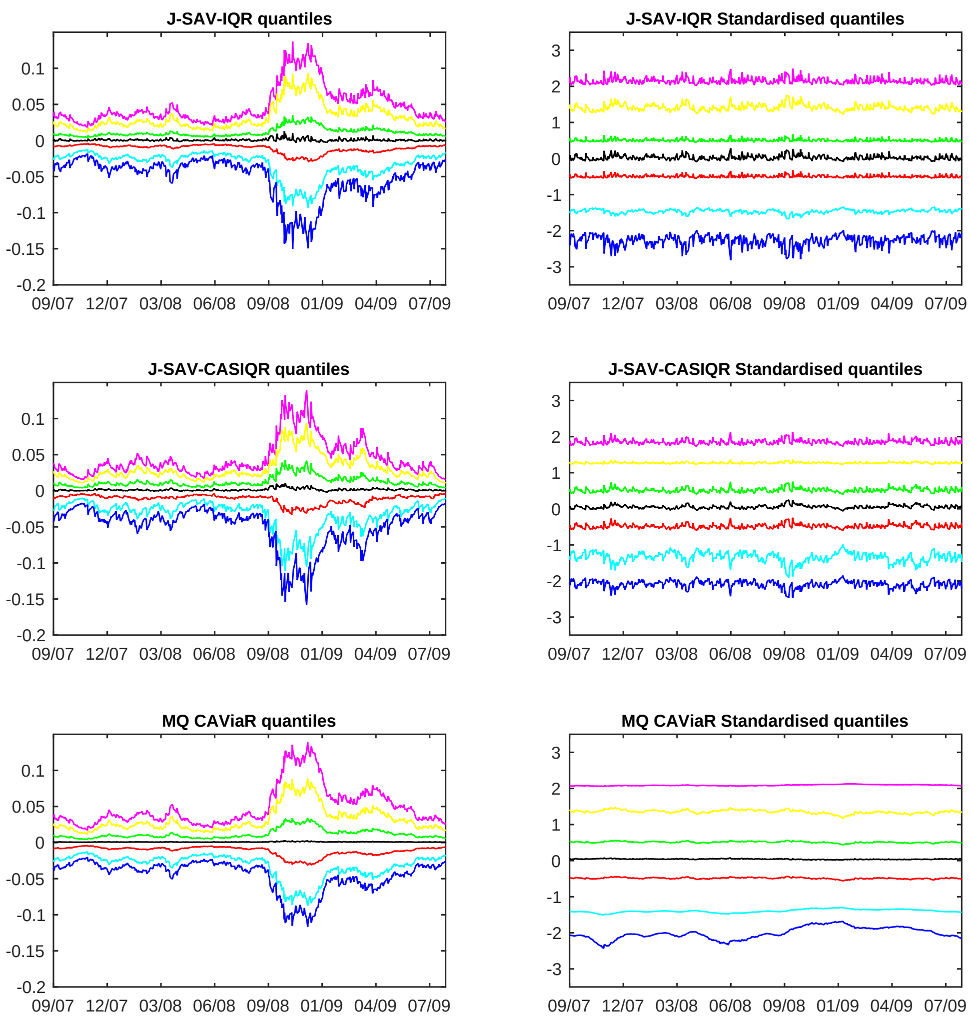


Figure A3: Quantile estimates for S&P500 from September $01, 2007$, to August $31, 2009$.

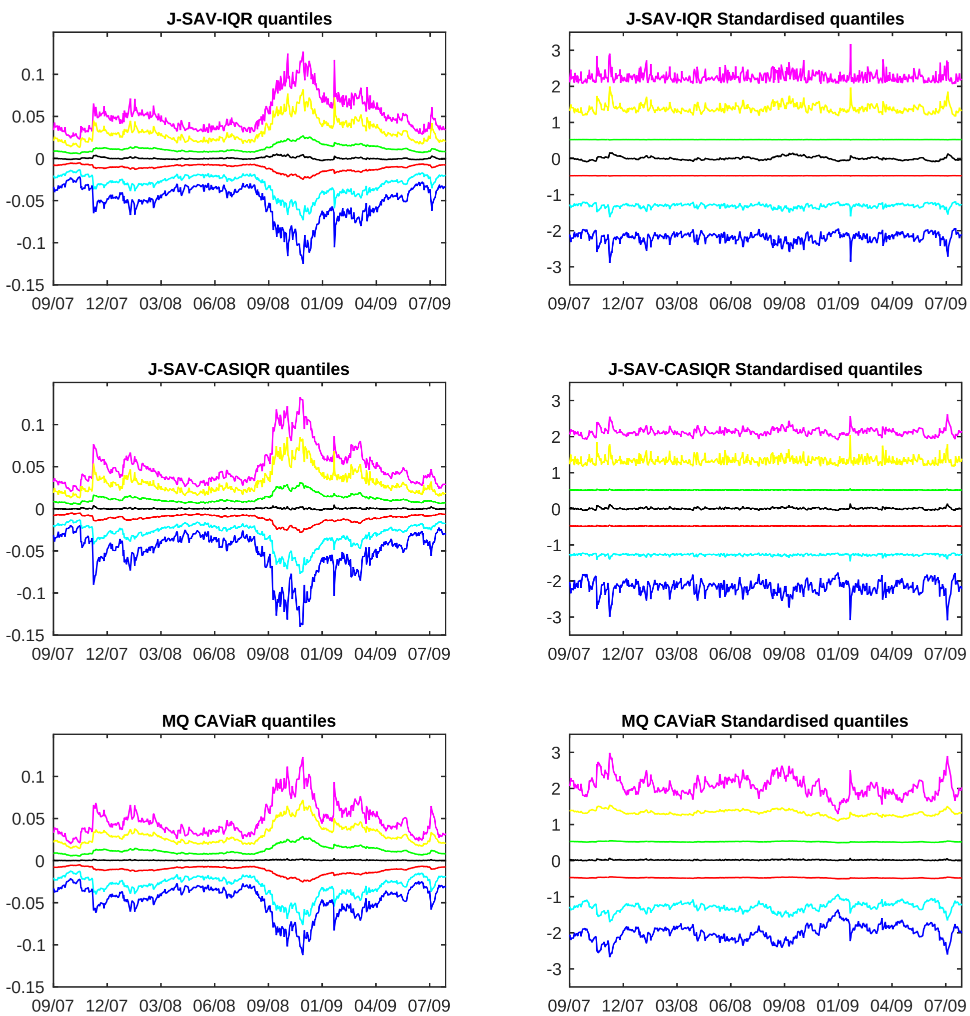


Figure A4: Quantile estimates for IBM from September 01, 2007, to August 31, 2009.

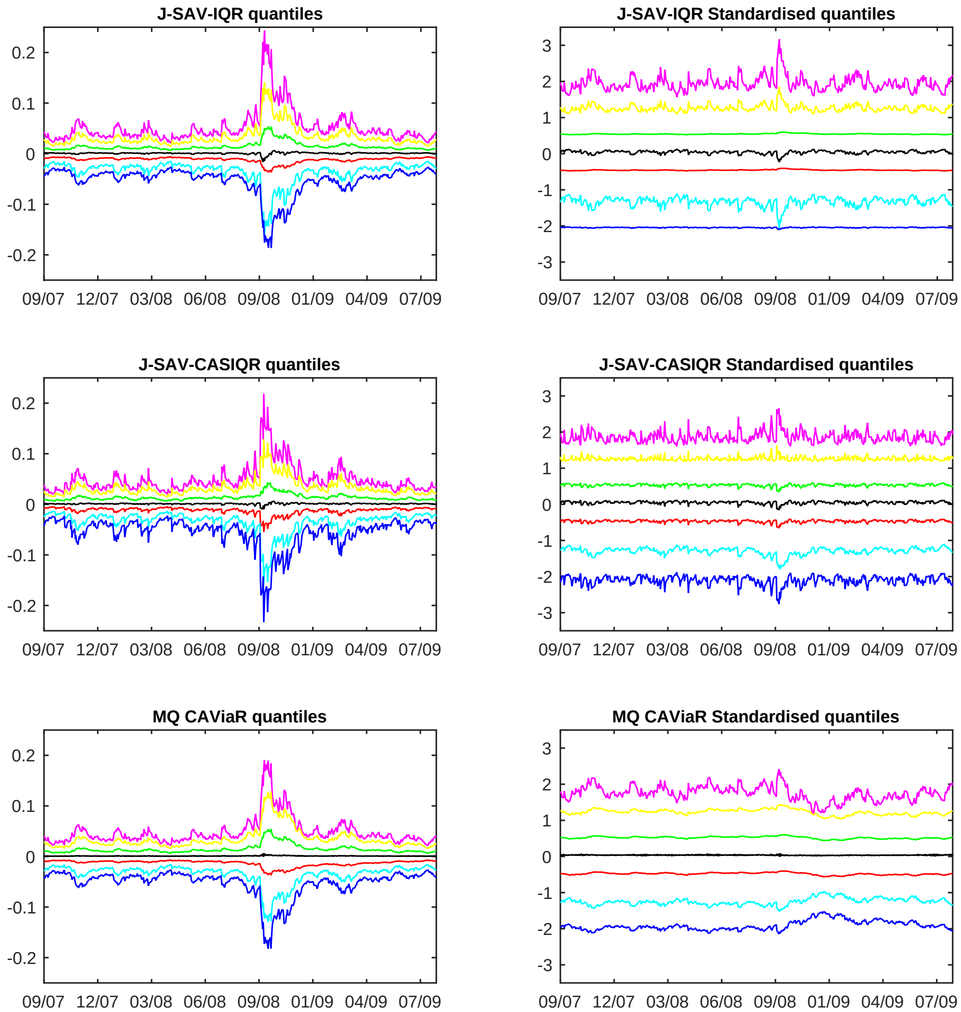


Figure A5: Quantile estimates for XOM from September 01, 2007, to August 31, 2009.

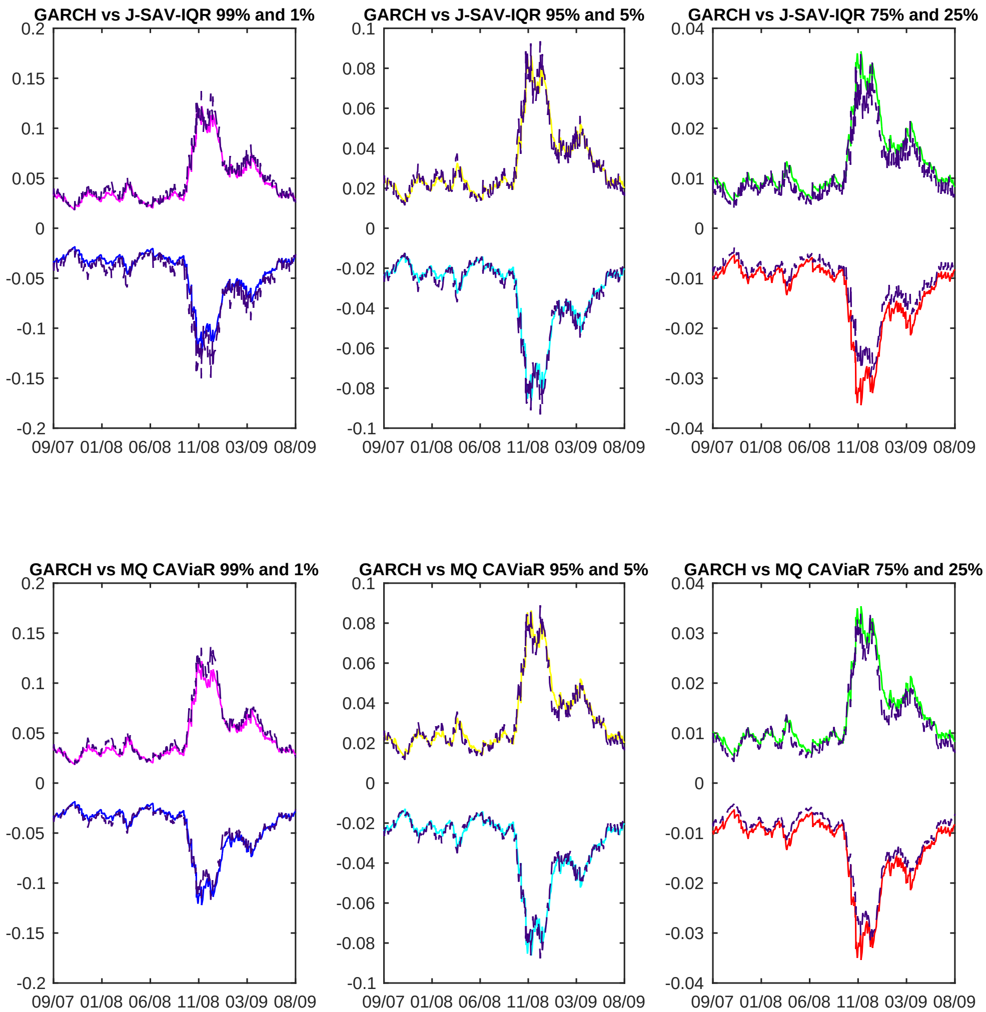


Figure A6: GARCH vs J-SAV-IQR (upper panel) and GARCH vs MQ CAViaR (lower panel) for S&P500 from September $01, 2007$, to August $31, 2009$.

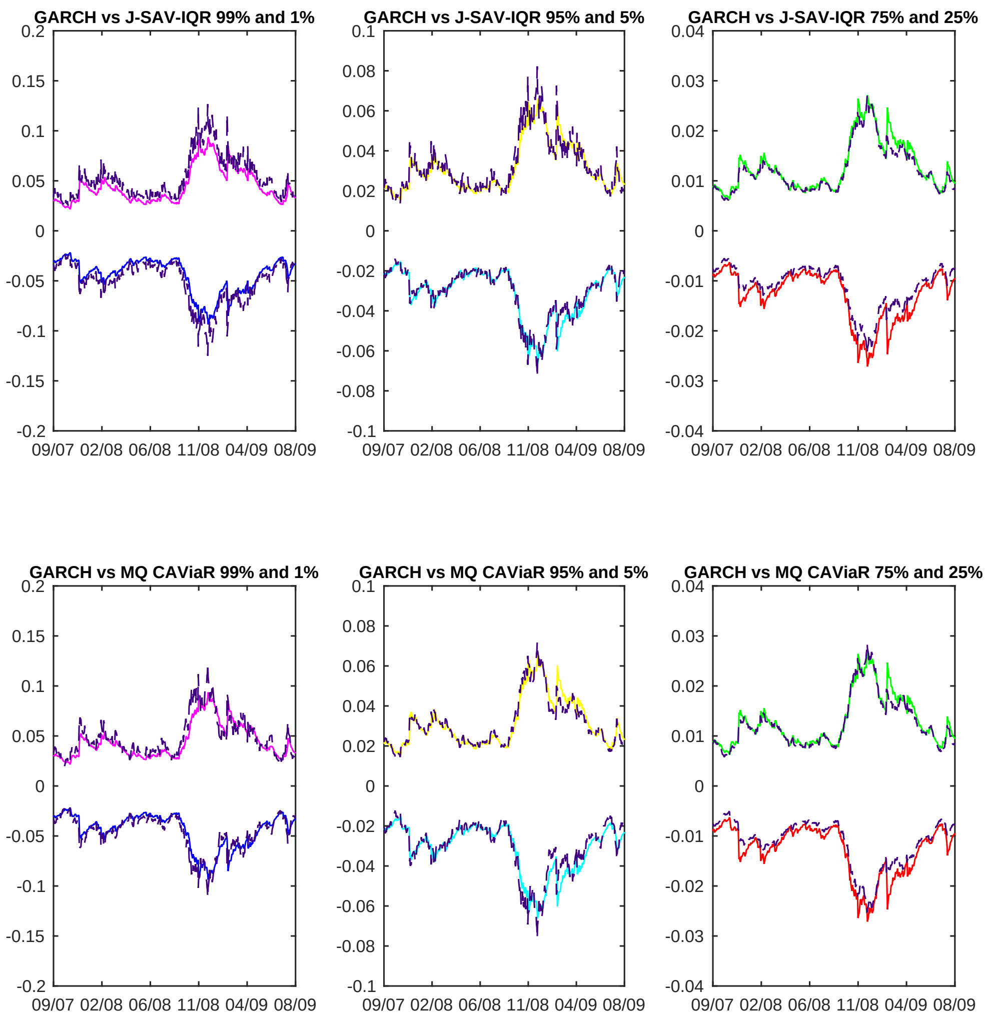


Figure A7: GARCH vs J-SAV-IQR (upper panel) and GARCH vs MQ CAViaR (lower panel) for IBM from September 01, 2007, to August 31, 2009.

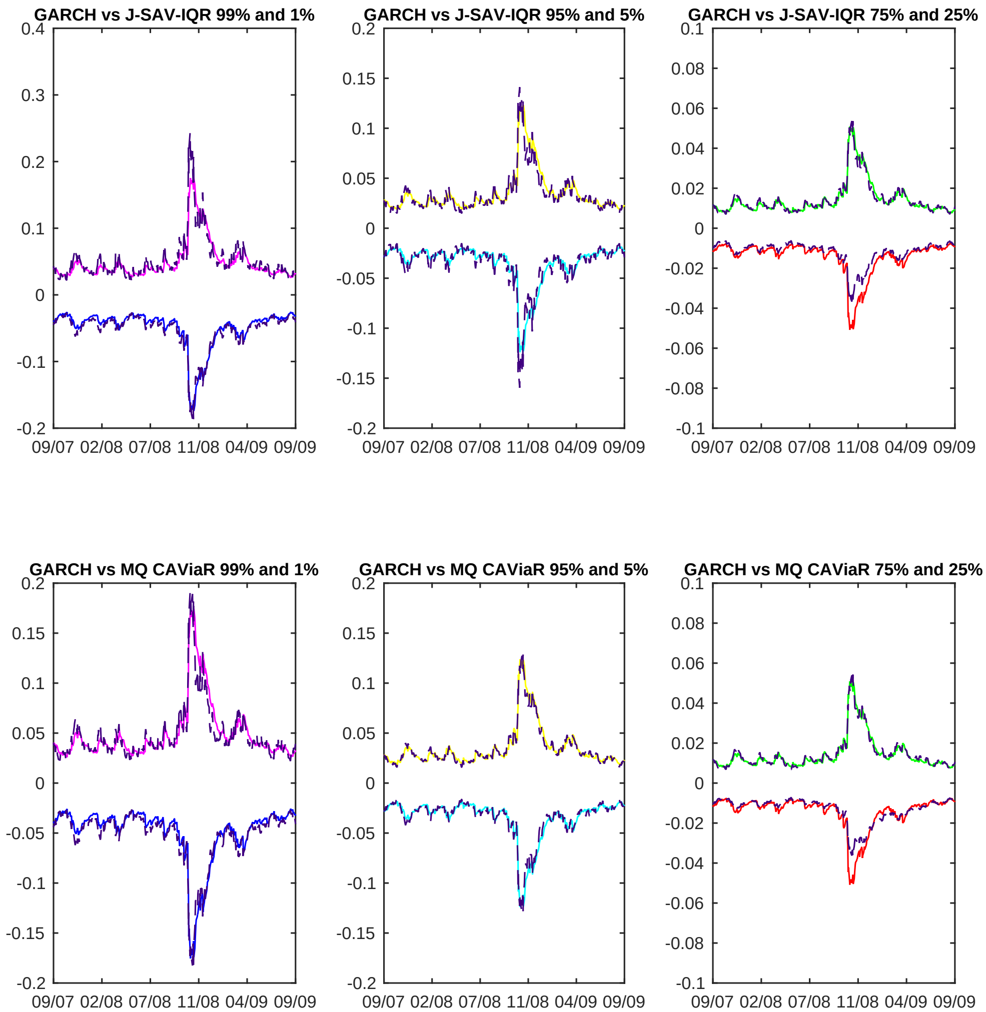
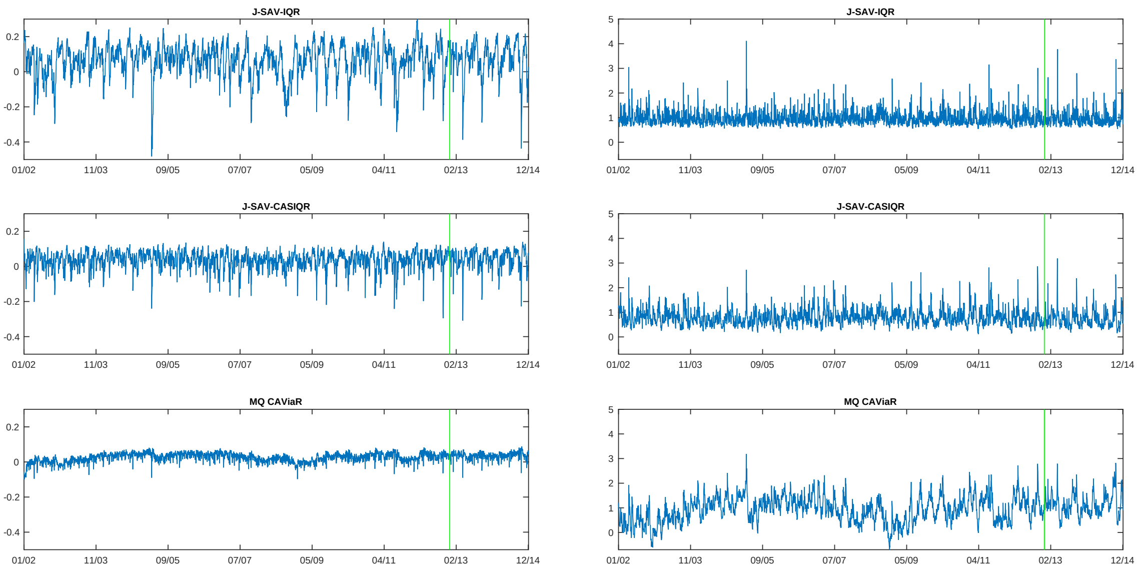


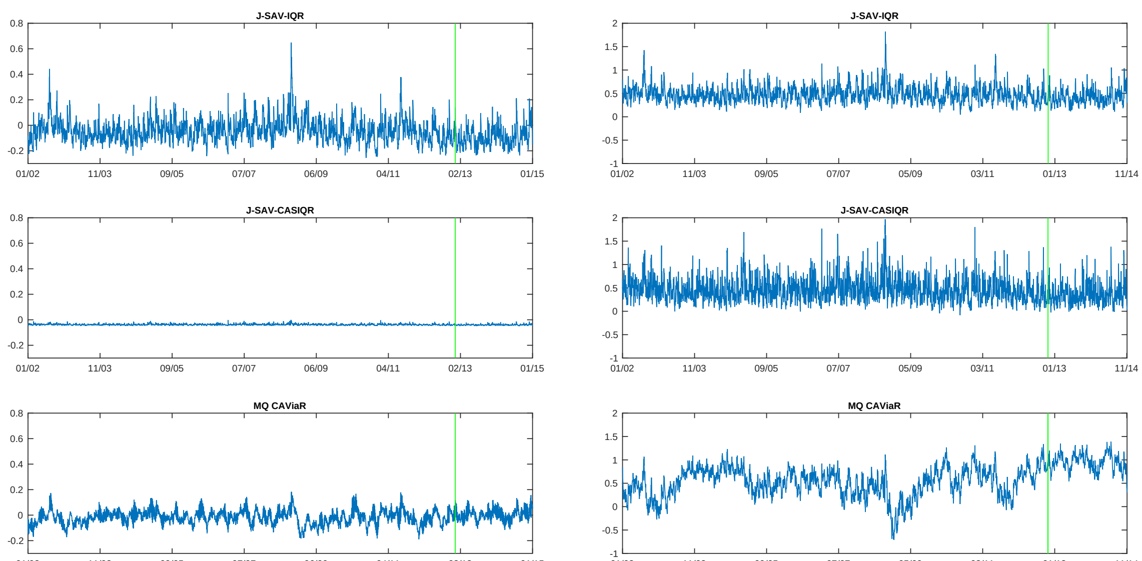
Figure A8: GARCH vs J-SAV-IQR (upper panel) and GARCH vs MQ CAViaR (lower panel) for XOM from September 01, 2007, to August 31, 2009.



(a) Skewnes

(b) Kurtosis

Figure A9: Quantile-based measures of conditional skewness and kurtosis for IBM



(a) Skewnes

(b) Kurtosis

Figure A10: Quantile-based measures of conditional skewness and kurtosis for XOM